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GUST: A General Unified Similarity Theory for the Calculation of Turbulent Fluxes in Numerical Weather Prediction Models for Unstable Conditions

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ABSTRACT

Flux-profile relations are presented for wind and temperature in the surface layer of the unstable atmospheric boundary layer. The flux-profile relation for wind is derived from the so-called 'KEYPS' equation, a quartic equation whose real, positive root is equal to the nondimensional wind shear. The analogous relation for potential temperature is a new cubic equation whose real, positive root is equal to the nondimensional temperature gradient (Helfand, 1984). It is assumed that the nondimensional humidity gradient is equal to that for temperature.

The nondimensional gradients reduce to conventional empirical relations for conditions of weak instability but, unlike the conventional relations, go to the 'free-convection' limit for conditions of strong instability.

The integrated forms of the flux-profile relations yield expressions which can be solved iteratively for the Obukhov length which, in turn, can be used to compute surface momentum, heat, and moisture fluxes. Although the flux-profile relations and their integrated forms have no evident closed-form solutions, rational fraction and asymptotic approximations are derived that are no more costly computationally than the expressions currently in use in numerical prediction models that use conventional flux-profile relations.

I. Introduction

Perhaps the most important practical problems in boundary layer meteorology are the determination of the turbulent fluxes of momentum, heat, and moisture and also the determination of the profiles of wind, temperature, and specific humidity near the earth's surface.

The similarity theory of Obukhov (1946) and Monin and Obukhov (1953, 1954) is the centerpiece of all such flux and profile considerations. Obukhov similarity theory shows that turbulent fluxes in the atmospheric surface layer can be uniquely determined from measurements of roughness length (z_0), ground temperature and humidity (θ_0, q_0), and temperature, wind speed, and humidity at an additional level $z = h \gg z_0$. The nondimensional bulk Richardson number R_B ($= g h \Delta\theta / \bar{\theta} U^2$) determines the scaled height h/L (L = Obukhov length) from which surface fluxes may be computed (g = acceleration due to gravity; $\bar{\theta}$ = reference potential temperature). This result is of considerable importance in numerical prediction models that use a balance of solar, long wave, sensible, latent, and soil heat fluxes to predict or diagnose surface temperature and humidity.

Obukhov similarity theory postulates the existence of universal functions φ_m and φ_h such that the nondimensional wind and temperature shears are given by

$$\frac{kz}{u_*} \frac{\partial U}{\partial z} = \varphi_m(z/L) \tag{1.1a}$$

and

$$\frac{kz}{b\theta_*} \frac{\partial \theta}{\partial z} = \varphi_h(z/L) \tag{1.1b}$$

in which k is the von Kármán constant and b is the neutral Prandtl number. A similar expression exists for humidity,

$$\frac{kz}{cq_*}\frac{\partial q}{\partial z} = \varphi_q\,(z/L)\,. \tag{1.1c}$$

(See the appendix for a definition of the symbols used in this report.)

Although not required by Obukhov similarity theory, it is usually assumed that $\varphi_q = \varphi_h$ and c = b.

No complete theory exists that predicts $\varphi_m(z/L)$ and $\varphi_h(z/L)$ over the entire stable and unstable range. Moreover, experimental data are still lacking in sufficient precision to specify interpolation formulae for φ_m and φ_h without equivocation. Accordingly, semi-empirical functions abound. Dyer (1974) and Yaglom (1977) have surveyed many such relations for stable and unstable cases. The relations for unstable cases that have probably enjoyed the greatest popularity in numerical models are those of Businger et al. (1971),

$$\varphi_m(z/L) = (1 - 15\,z/L)^{-1/4} \tag{1.2a}$$

$$\varphi_h(z/L) = (1 - 9\,z/L)^{-1/2} \quad ; \quad k = 0.35 \tag{1.2b}$$

and Dyer and Hicks (1970),

$$\varphi_m(z/L) = (1 - 16\,z/L)^{-1/4} \tag{1.3a}$$

$$\varphi_h(z/L) = (1 - 16\,z/L)^{-1/2} \quad ; \quad k = 0.41\,. \tag{1.3b}$$

The integrated forms of (1.2,3), valid for $-z/L \geq 0$, [see (3.5)] can be expressed in closed-form, although the results are somewhat complicated. It is probably the closed-form nature of the integrated expressions that accounts for their popularity among numerical modelers.

None of these expressions, however, has the correct 'free-convective' asymptotic form for $-z/L \rightarrow \infty$, first predicted by Prandtl (1932) and later by Obukhov (1946), Priestly (1954), and Kazansky and Monin (1958). As the free-convection limit is approached, the surface heat flux becomes increasingly independent of U and ultimately becomes proportional to $\Delta\Theta^{3/2}$. There are two ways by which the limit may be approached: either by increasing z or by decreasing u_* to $u_* \longrightarrow 0$.

For the free-convection limit to hold, dimensional analysis shows that $\partial U/\partial z$ and $\partial\theta/\partial z$ must have the limiting form

$$\frac{\partial U}{\partial z} \sim z^{-4/3} \quad ; \quad \varphi_m \sim (-z/L)^{-1/3} \tag{1.4a}$$

$$\frac{\partial \theta}{\partial z} \sim z^{-4/3} \quad ; \quad \varphi_h \sim (-z/L)^{-1/3} \tag{1.4b}$$

(see, e.g., Monin and Yaglom, 1971 and Lumley and Panofsky, 1964), which apparently conflicts with (1.2,3) for which

$$\frac{\partial U}{\partial z} \sim z^{-5/4} \quad ; \quad \varphi_m \sim (-z/L)^{-1/4} \tag{1.5a}$$

$$\frac{\partial \theta}{\partial z} \sim z^{-3/2} \quad ; \quad \varphi_h \sim (-z/L)^{-1/2}. \tag{1.5b}$$

The nature of this discrepancy has not been resolved. If the free-convection results are valid, then it may be that the sparseness of data points for $-z/L$ greater than ≈ 2 (see Businger, et al., 1971) is the source of the difficulty. Indeed, Carl et al. (1973) argue that a composite of tower data suggests that

$$\varphi_m(z/L) = (1 - 16z/L)^{-1/3} \tag{1.6}$$

for $-z/L \geq 0$. Eq. (1.6) obviously reduces to the free-convection limit $(-z/L)^{-1/3}$ as $-z/L \rightarrow \infty$.

The so-called KEYPS profile is an interpolation formula for φ_m. 'KEYPS' is constructed from the initials of the Western researchers who independently devised the profile after Obukhov (1946) first proposed it. The KEYPS profile is computed from the solution of the following defining quartic relation,

$$\varphi_m^4 - \gamma_m z/L \; \varphi_m^3 - 1 = 0 \tag{1.7a}$$

in which γ_m is a constant. Recently, Helfand (1984) has proposed a cubic equation to supplement (1.7a)

$$\varphi_h^2 - \gamma_h z/L \; \varphi_h^3 - 1 = 0 \tag{1.7b}$$

to account for the universal temperature profile.

Our purpose in this report is not to attempt to show that (1.7) represent observational data with greater fidelity than the Businger or Dyer profiles. We shall be content, however, to show that: (a), (1.7) reduce to the Businger-Dyer relationships for mild instability and the free-convection limit for strong instability ($-z/L \gg 1$); and (b), although the defining quartic and cubic relations appear to be computationally impractical, they can be utilized in numerical models at least as efficiently as the Businger-Dyer relations.

For the remainder of this report, we shall refer to the KEYPS and Helfand relations collectively as the GUST (= General Unified Similarity Theory) relations.

2. Solutions to the flux-profile relations

In this section, we will show that the GUST relations tend toward the Businger-Dyer relations for small values of $-z/L$ and the free-convection relations for large values of $-z/L$. We will then present simple methods for computing φ_m and φ_h for the entire unstable regime $0 \leq -z/L < \infty$.

As a primary requirement, we must determine if each of the GUST relations,

$$\varphi_m^4 - \gamma_m z/L \; \varphi_m^3 - 1 = 0 \tag{2.1a}$$

$$\varphi_h^2 - \gamma_h z/L \; \varphi_h^3 - 1 = 0 \tag{2.1b}$$

rewritten more conveniently with $\eta = -\gamma z/L$,

$$\varphi_m^4 + \eta_m \varphi_m^3 - 1 = 0 \tag{2.2a}$$

$$\varphi_h^2 + \eta_h \varphi_h^3 - 1 = 0 \tag{2.2b}$$

has one, and no more than one, positive (physical) root for each value of $\eta \geq 0$. Descartes' 'rule of signs' (Korn and Korn, 1965) states the number of positive roots of a polynomial cannot exceed the number of sign changes of the polynomial's coefficients. Since the GUST relations have only one sign change, there can be no more than one positive root for each equation. We shall show how the roots can be computed efficiently later in this section.

To show that the GUST relations reduce to the standard flux-profile relations for small values of η , we recast (2.2) into the forms

$$\varphi_m = (1 + \eta_m / \varphi_m)^{-1/4} \tag{2.3a}$$

$$\varphi_h = (1 + \eta_h \varphi_h)^{-1/2} . \tag{2.3b}$$

For the case of weak convection, $\eta \to 0$ and $\varphi_{m,h} \to 1$; therefore, for weak instability,

$$\varphi_m \approx (1 + \eta_m)^{-1/4} \tag{2.4a}$$

$$\varphi_h \approx (1 + \eta_h)^{-1/2} . \tag{2.4b}$$

The exponents -1/2 and -1/4 agree with most of the semi-empirical flux-profile relations. We note, however, that series expansions of the GUST profile relations agree with conventional profiles only to orders $O(1)$ and $O(\eta)$, and differ at $O(\eta^2)$ and higher orders.

To find the limiting forms of φ_m and φ_h for strong convection, we rewrite (2.2) as

$$\varphi_m = \eta^{-1/3} (1 + \varphi_m/\eta)^{-1/3} \tag{2.5a}$$

$$\varphi_h = \eta^{-1/3}\left(1 + 1/\eta\varphi_h\right)^{-1/3} \tag{2.5b}$$

(hereafter, for simplicity, the subscripts on η will be omitted). As $\eta \to \infty$, the term φ_m/η in (2.5b) vanishes and $\varphi_m \sim \eta^{-1/3}$. As $\eta \to \infty$, we shall tentatively assume that $\eta\varphi_h \to \infty$, in which case (2.5a) asymptotically becomes $\varphi_h \sim \eta^{-1/3}$. We see that the tentative assumption, $\eta\varphi_h \to \infty$, is consistent with $\varphi_h \sim \eta^{-1/3}$, since $\eta\varphi_h \sim \eta^{2/3} \to \infty$. The limiting relations $\varphi_{m,h} \sim \eta^{-1/3}$ are sometimes referred to as 'strict' free-convection scaling. (Tennekes, 1973; Zilitinkevitch, 1970).

In addition to their yielding asymptotic relations for $\eta \to 0$ and $\eta \to \infty$, (2.3) and (2.5) lead to efficient iteration formulas for evaluating φ_m and φ_h, that is,

$$\varphi_m^{(j+1)} = \left(1 + \eta / \varphi_m^{(j)}\right)^{-1/4} \tag{2.6a}$$

$$\varphi_h^{(j+1)} = \left(1 + \eta\,\varphi_h^{(j)}\right)^{-1/2} \tag{2.6b}$$

and

$$\varphi_m^{(j+1)} = \eta^{-1/3}\left(1 + \varphi_m^{(j)}/\eta\right)^{-1/3} \tag{2.7a}$$

$$\varphi_h^{(j+1)} = \eta^{-1/3}\left(1 + 1/\eta\,\varphi_h^{(j)}\right)^{-1/3} \tag{2.7b}$$

where (j) and (j+1) denote the j and j+1 iterates, beginning with $\varphi_{m,h}^{(0)} = 1$. Computational experiments indicate that (2.6) converge more quickly than (2.7) for $\eta \leq 1$ and that (2.7) converge more quickly than (2.6) for $\eta > 1$. For a fractional error of 10^{-3}, no more than five iterations are required; for a fractional error of 10^{-8}, fewer than a dozen iterations suffice. Values of $\varphi_{m,h}$ computed from the iteration relations are presented in Table 1.

An alternative method for generating a table of $\varphi_{m,h}$ consists of converting the quartic and cubic algebraic GUST equations into ordinary differential equations. Differentiating the quartic equation for φ_m yields

Table 1. The nondimensional universal functions φ_m and φ_h as a function of η.

η	φ_m	φ_h
0.001	0.99975	0.99950
0.005	0.99875	0.99752
0.01	0.99751	0.99506
0.05	0.98773	0.97645
0.10	0.97591	0.95540
0.50	0.89498	0.83929
1.0	0.81917	0.75488
2.5	0.68002	0.62477
5.0	0.56432	0.52517
10.0	0.45729	0.43310
25.0	0.34046	0.32917
50.0	0.27095	0.26494
100.0	0.21529	0.21216
500.0	0.12598	0.12533
1000.0	0.099997	0.099667
5000.0	0.058480	0.058414

$$\left(4\varphi_m^3 + 3\eta\varphi_m^2\right)\frac{d\varphi_m}{d\eta} + \varphi_m^3 = 0 \tag{2.8}$$

with $\varphi_m(0) = 1$. This procedure converts the quartic algebraic equation into an initial value problem that can be 'marched' from $\eta = 0$ to the largest desired value of η. In practice, $\eta \approx 5$ represents a reasonable upper limit. Beyond $\eta = 5$, simple asymptotic algebraic formulas work quite well.

Unlike the Businger or Dyer flux-profile laws, (2.7) and (2.8) suggest that there are no closed-form solutions for $\varphi_{m,h}$. Rational functions, however, can be used to approximate $\varphi_{m,h}$ effectively. Given the function y(x), the rational function $Y_{L,M}(x)$, defined by

$$Y_{L,M}(x) = \frac{a_0 + a_1 x + a_2 x^2 + \ldots a_L x^L}{1 + b_1 x + b_2 x^2 + \ldots b_M x^M} \tag{2.9}$$

can be used to collocate with $y_j = f(x_j)$ at j = 1, 2, ... N = L + M + 1 values of x_j and y_j, since there are L + M + 1 undetermined coefficients in the numerator and denominator of (2.9). The substitution of N values of (x_j, y_j) into (2.9) leads to a N x N system of equations in which $\{a_0, a_1, a_2, \ldots a_L, b_1, b_2, \ldots b_M\}$ is the solution vector. In order to approximate $\varphi_{m,h}(\eta)$ in the range $0 \leq \eta \leq 5$, it is sufficient to choose the seven collocation points $\eta_j = \{0, 0.01, 0.1, 0.5, 1.0, 2.5, 5.0\}$. The choice of seven points allows a cubic polynomial in both the numerator and the denominator of (2.9). Solving the 7 x 7 systems leads to

$$\varphi_m \approx \frac{1 + 0.09570753 6\eta + 0.12113565\eta^2 + 0.0127995 57\eta^3}{1 + 0.34574687\eta + 0.11323529\eta^2 + 0.042095516\eta^3} \tag{2.10a}$$

and

$$\varphi_h \approx \frac{1 + 2.1809521\eta + 0.90291804\eta^2 + 0.040083289\eta^3}{1 + 2.6808844\eta + 0.11323529\eta^2 + 0.16232657\eta^3}. \tag{2.10b}$$

Table 2 shows the values computed from (2.10) compared to the exact values computed from the interation formulas. The interpolation **error is relatively small and is of no practical significance.**

For values of η greater than $\approx$ 5.0, simple three-term asymptotic expressions can be invoked to approximate $\varphi_{m,h}$. As was shown in the discussion following (2.5), $\varphi_{m,h} \sim \eta^{-1/3}$ as $\eta \to \infty$. The term $\eta^{-1/3}$ is the leading term of asymptotic expressions for $\varphi_{m,h}$. Additional terms can be calculated from the iterative expressions for $\varphi_{m,h}$. Writing (2.5) again, we have

$$\varphi_m = \eta^{-1/3}\left(1 + \varphi_m/\eta\right)^{-1/3} \tag{2.5a}$$

$$\varphi_h = \eta^{-1/3}\left(1 + 1/\eta\varphi_h\right)^{-1/3}. \tag{2.5b}$$

Beginning with (2.5a), we use $\varphi_m \sim \eta^{-1/3}$ as the leading term in the asymptotic expansion for φ_m. That is,

$$\varphi_m^{(0)} = \eta^{-1/3}. \tag{2.11}$$

From (2.5a), we have

$$\varphi_m = \eta^{-1/3}\left(1 + \varphi_m/\eta\right)^{-1/3} = \eta^{-1/3}\left[1 - \frac{1}{3}\varphi_m/\eta + \frac{2}{9}\left(\varphi_m/\eta\right)^2 + \ldots\right] \tag{2.12}$$

Substituting $\varphi_m^{(0)}$ into the first two terms of (2.12) gives $\varphi_m^{(1)}$,

$$\varphi_m^{(1)} = \eta^{-\frac{1}{3}}\left(1 - \frac{1}{3}\varphi_m^{(0)}/\eta\right) = \eta^{-1/3}\left(1 - \frac{1}{3}\eta^{-4/3}\right) \tag{2.13}$$

Similarly, substituting $\varphi_m^{(1)}$ into the first three terms of (2.12) and retaining terms of order no higher than $\eta^{-8/3}$, we have

$$\varphi_m^{(3)} = \eta^{-1/3}\left(1 - \frac{1}{3}\eta^{-4/3} + \frac{1}{3}\eta^{-8/3}\right) \tag{2.14a}$$

or

$$\varphi_m^{(3)} = q\left(1 - \frac{1}{3}q^4 + \frac{1}{3}q^8\right) ; q = \eta^{-1/3}. \tag{2.14b}$$

Table 2. Comparison of the exact values of φ_m and φ_h with the approximate values computed from Eqs. (2.10).

η	φ_m^{exact}	$\varphi_m^{Eq. (2.10a)}$	φ_h^{exact}	$\varphi_h^{Eq. (2.10b)}$
0.001	0.99975	0.99975	0.99950	0.99950
0.05	0.98773	0.98773	0.97645	0.97648
0.25	0.94290	0.94290	0.90321	0.90331
2.0	0.71667	0.71673	0.65730	0.65738
3.0	0.64951	0.64941	0.59819	0.59822
4.0	0.60123	0.60100	0.55669	0.55668

Table 3, below, compares (2.14) with the exact values of φ_m.

Table 3. Comparison of the exact values of φ_m with the approximate values computed from Eq. (2.14)

η	φ_m^{exact}	$\varphi_m^{(3)}$
0	1.0	∞
2	0.71667	0.73037
5	0.56433	0.56467
10	0.45729	0.45731
25	0.34046	0.34046
50	0.27095	0.27095

As can be seen, $\varphi_m^{(3)}$ works sufficiently well for $\eta \geq 5$ that we can, for practical purposes, replace φ_m exact with its $\varphi_m^{(3)}$ asymptotic approximation. A similar computation can be carried out for φ_h. The results are,

$$\varphi_h = \eta^{-1/3}\left[1 - \tfrac{1}{3}(\eta\varphi_h)^{-1} + \tfrac{2}{9}(\eta\varphi_h)^{-2} + \ldots\right] \qquad (2.15)$$

$$\varphi_h = \eta^{-1/3}\left[1 - \tfrac{1}{3}\eta^{-2/3} + \tfrac{1}{9}\eta^{-4/3} + O(\eta^{-3})\right] \qquad (2.16)$$

$$\varphi_h^{(3)} = g\left(1 - \tfrac{1}{3}g^2 + \tfrac{1}{9}g^4\right). \qquad (2.17)$$

Table 4 compares (2.17) with the exact values of φ_h.

Table 4. Comparison of the exact values of φ_h with the approximate values computed from Eq. (2.27)

η	φ_h^{exact}	$\varphi_h^{(3)}$
0	1.0000	∞
2	0.65730	0.66203
5	0.52517	0.52574
10	0.43311	0.43322
25	0.32917	0.32918
50	0.26494	0.26494

Since the terms for φ_m go as $1, g^4, g^8$, ... , while the terms for φ_h go as $1, g^2, g^4$, ... , the φ_m function approaches the free-convective limit more quickly than the φ_h function.

3. Integrated forms of the flux-profile relations

Integration of the flux-profile relations yields profiles of wind and potential temperature. It is convenient and also standard practice to integrate between the roughness height z_0 and an arbitrary height $z = h >> z_0$ using the dummy variable $\xi'(= z'/L)$,

$$U(z) = \frac{u_*}{k}\int_{z_0}^{z}\frac{dz'}{z'}\,\varphi_m(z'/L) = \frac{u_*}{k}\int_{\xi_0}^{\xi}\frac{d\xi'}{\xi'}\,\varphi_m(\xi') \qquad (3.1a)$$

$$\Theta(z) - \Theta(z_0) = \frac{b\,\theta_*}{k} \int_{\xi_0}^{\xi} d\xi' \, \frac{\varphi_h(\xi')}{\xi'} . \tag{3.1b}$$

The roughness height is conventionally taken as the air-ground interface level at which a surface energy balance is computed in a numerical prediction model. The possible distinction between the momentum roughness height and the heat (and vapor) roughness height is usually ignored by numerical modelers (see Brutsaert, 1982, however, for a congently argued opposing view).

The wind speed and potential temperature at a given height $z=h$ can be calculated from u_*, θ_*, L, z_0, $\theta(z_0)$, and the integrated flux-profile relations. It is the inverse problem, however, that is usually of greater interest to numerical modelers. Given $U(h)$, $\Delta\theta = \theta(h) - \theta(z_0)$, and z_0, the inverse problem is to determine L from which U_* and θ_* can be calculated. The surface momentum and heat fluxes follow from $-\rho u_*^2$ and $-\rho C_p u_* \theta_*$ in which ρ is the density of air and C_p is the specific heat of air at constant pressure.

The Obukhov length may be computed by first noting that since

$$L = \frac{\bar{\theta}}{kg} \frac{u_*^2}{\theta_*} \tag{3.2}$$

then

$$R_B = \frac{h}{L} \, b \, \frac{F_h}{F_m^2} \tag{3.3}$$

in which R_B is the bulk Richardson number ($= \frac{g}{\bar{\theta}} \frac{h\Delta\theta}{U^2}$), and F_m and F_h are the integrated flux-profile relations,

$$F_m = F_m\left(\frac{h}{L};\frac{z_o}{L}\right) = \int_{\zeta_o}^{\zeta} \frac{d\zeta'}{\zeta'}\,\varphi_m(\zeta') \tag{3.4a}$$

$$F_h = F_h\left(\frac{h}{L};\frac{z_o}{L}\right) = \int_{\zeta_o}^{\zeta} \frac{d\zeta'}{\zeta'}\,\varphi_h(\zeta'). \tag{3.4b}$$

Solving for $\zeta = h/L = R_B F_m^2/b F_h$ requires iteration since $F_m{}^2/F_h$ is a non-linear function of ζ.

The functions F_m and F_h can be somewhat complicated. For example, the Businger relations $\varphi_m = (1-15\,\zeta)^{-1/4}$; $\varphi_h = (1-9\zeta)^{-1/2}$ yield (Long and Shaffer, 1975),

$$F_m = \ln\left[\frac{(R_h-1)(R_o+1)}{(R_h+1)(R_o-1)}\right] + \tan^{-1} R_m - \tan^{-1} R_o \tag{3.5a}$$

$$F_h = \ln\left[\frac{(Q_h-1)(Q_o+1)}{(Q_h+1)(Q_o-1)}\right] \tag{3.5b}$$

in which $R_h = (1-15\,\zeta)^{1/4}$, $R_o = (1-15\,\zeta_o)^{1/4}$, $Q_h = (1-9\zeta)^{1/2}$, $Q_o = (1-9\zeta_o)^{1/2}$.
For the GUST relations, no known closed-form expressions for F_m and F_h exist. In what follows, however, we shall show how F_m and F_h can be approximated rather closely by simple expressions that are similar in form to those derived for φ_m and φ_h.

Although there are no apparent analytical expressions relating ζ to F_m and F_h, there are closed-form expressions relating F_m and F_h to φ_m and φ_h. To derive them, we begin by writing F_m and F_h in what has become standard form; that is,

$$F_m = \ln\left(\frac{h}{z_0}\right) - \Psi_m(\xi, \xi_0) \qquad (3.6a)$$

$$F_h = \ln\left(\frac{h}{z_0}\right) - \Psi_h(\xi, \xi_0) \qquad (3.6b)$$

in which

$$\Psi_m = \int_{\xi_0}^{\xi} d\xi' \; \frac{1 - \varphi_m(\xi')}{\xi'} \qquad (3.7a)$$

and

$$\Psi_h = \int_{\xi_0}^{\xi} d\xi' \; \frac{1 - \varphi_h(\xi')}{\xi'} \; . \qquad (3.7b)$$

The expressions involving $\Psi_{m,h}$ preserve the logarithmic forms of the profile laws that are obscured by (3.4). Since for conditions close to neutrality the profile laws reduce to the log-laws,

$$U(h) = \frac{u_*}{k} \ln\left(\frac{h}{z_0}\right) \qquad (3.8a)$$

$$\Delta\theta = \frac{b\theta_*}{k} \ln\left(\frac{h}{z_0}\right), \qquad (3.8b)$$

we may regard $\Psi_{m,h}$ as correction terms to the logarithmic profile laws that account for diabatic conditions. For near-neutral conditions, $\Psi_{m,h} \rightarrow 0$.

To compute $\Psi_{m,h}$ as a function of $\varphi_{m,h}$, we differentiate $\Psi_{m,h}$ with respect to η,

$$\frac{d\psi_{m,h}}{d\eta} = \frac{1-\varphi_{m,h}}{\eta} \tag{3.9}$$

in which the η in the denominator may be eliminated by invoking the defining GUST relations [(2.2)]. Thus, for φ_h we have,

$$\frac{d\psi_h}{d\eta} = \frac{\varphi_h^3}{1+\varphi_h}. \tag{3.10}$$

From (2.2) we also have,

$$\frac{d\varphi_h}{d\eta} = \frac{1}{\varphi_h^2} - \frac{3}{\varphi_h^4} \tag{3.11}$$

from which we get, after reducing to partial fractions,

$$\frac{d\psi_h}{d\varphi_h} = \frac{d\psi_h/d\eta}{d\varphi_h/d\eta} = \frac{\varphi_h}{1+\varphi_h} - \frac{3}{\varphi_h(1+\varphi_h)}. \tag{3.12}$$

By numerical integration or by 'marching' methods we can create a table of ψ_h as a function of φ_h or we may simply integrate (3.12) analytically to get

$$\psi_h(\varphi_h) = (\varphi_h - 1) - 3\ln\varphi_h + 2\ln\left(\frac{1+\varphi_h}{2}\right). \tag{3.13a}$$

Similarly, for $\psi_m(\varphi_m)$, we have, assuming $\eta_o \ll 1$

$$\psi_m(\varphi_m) = 1 - \varphi_m - 3\ln\varphi_m + 2\ln\left(\frac{1+\varphi_m}{2}\right) + \ln\left(\frac{1+\varphi_m^2}{2}\right) + 2\tan^{-1}\varphi_m - \pi/2 \tag{3.13b}$$

(Paulson, 1970). If we elect to use rational fraction interpolation for $0 \leq \eta \leq 5$, we can combine (3.13) with Table 1 to force collocation at $\{0, 0.01, 0.1, 0.5, 1.0, 2.5, 5.0\}$.

This calculation yields

$$\psi_m \approx \frac{0.24994850\,\eta + 0.98087895\,\eta^2 + 0.096758127\,\eta^3}{1 + 4.1088095\,\eta + 1.1231890\,\eta^2 + 0.035316531\,\eta^3} \tag{3.14a}$$

$$\psi_h \approx \frac{0.49992847\,\eta + 0.62387737\,\eta^2 + 0.093432486\,\eta^3}{1 + 1.8700867\,\eta + 0.70579463\,\eta^2 + 0.036313720\,\eta^3}\,. \tag{3.14b}$$

A brief table for $\psi_{m,h}$ approx is given below. As with $\varphi_{m,h}$ approx, interpolation errors are negligible.

For $\eta \geq 5$, we can derive a pair of relations for $\psi_{m,h}$ similar in form to those for $\varphi_{m,h}$. To begin, we rewrite $\psi_{m,h}$ as

$$\psi_{m,h}(\eta) = \psi_{m,h}(\eta = a) + \int_a^\eta d\eta' \, \frac{1 - \varphi_{m,h}(\eta')}{\eta'}\,. \tag{3.15}$$

By setting a = 5 we can expect that the integral in (3.15) will be approximated closely by using the asymptotic relations for $\varphi_{m,h}$ given by (2.14) and (2.16). Since ψ_m (5) = 0.700734, we have

$$\psi_m(\eta \geq 5\,;\, a = 5) \approx 0.700734 + I_m(\eta) + C_m^a \tag{3.16}$$

where

$$I_m(\eta) = 3\eta^{-1/3} - \frac{1}{5}\eta^{-5/3} + \frac{1}{9}\eta^{-9/3} - \ln\eta \tag{3.17}$$

and C_m^a is a 'constant' of integration that depends upon a,

Table 5. Comparison of the exact values of $\psi_{m,h}$ with the approximate values computed from Eqs. (3.14)

η	ψ_m^{exact}	$\psi_m^{Eq.(3.14a)}$	ψ_h^{exact}	$\psi_h^{Eq.(3.14b)}$
0.001	0.00024796	0.00024795	0.00049968	0.00049969
0.05	0.012384	0.012383	0.024255	0.024257
0.25	0.059724	0.059727	0.10940	0.10940
2.0	0.37144	0.37144	0.54023	0.54022
3.0	0.49985	0.49983	0.69118	0.69120
4.0	0.60762	0.60761	0.81276	0.81284

$$C_m^a = -3a^{-1/3} + \frac{1}{5}a^{-5/3} - \frac{1}{9}a^{-9/3} + \ln a . \tag{3.18}$$

Substitution leads to

$$\psi_m^{exact}(a=5) + C_m^{a=5} = -2.65032 \tag{3.19}$$

so that,

$$\psi_m(\eta \geq 5) \simeq \ln \eta + 3z\left(1 - \frac{1}{15}z^4 + \frac{1}{27}z^8\right) - 2.65032. \tag{3.20}$$

Let us compare (3.20) with the analytical expression for ψ_m. For large values of η, the analytical result becomes

$$\lim_{\eta \to \infty} \psi_m(\eta) \sim \left(1 - 3\ln 2 - \frac{\pi}{2}\right) + \ln \eta + 3z \tag{3.21}$$

since $\phi_m \to \eta^{-1/3}$ for $\eta \to \infty$. The constant term in (3.21) equals −2.65024 and represents the limiting value of ψ_m(a) + C_m^a as $a \to \infty$. The difference between (3.19) in which a = 5 and the limiting value is of no practical significance. Comparisons between (3.20) and the exact values of are given below.

Table 6. Comparison of the exact values of Ψ_m with the approximate values computed from Eq. (3.20)

η	Ψ_m^{exact}	$\Psi_m^{Eq.(3.20)}$
5	0.70073	0.70073
6	0.78287	0.78282
8	0.92316	0.92309
10	1.0406	1.0405
25	1.5937	1.5936
50	2.0758	2.0757

The computation of $\Psi_h(\eta \geq 5)$ proceeds along similar lines. Using $\Psi_h(5)=$ 0.91520 and the relations (2.16) and (3.15), we get,

$$\Psi_h(\eta \geq 5) \approx \ln \eta + 3q\left(1 - \frac{1}{9}q^2 + \frac{1}{45}q^4\right) + \Psi_h(a=5) + C_h^a \qquad (3.22)$$

in which,

$$\Psi_h^{exact}(a=5) + C_h^{a=5} = -2.38654 \qquad (3.23)$$

The limiting form of the analytical expression for $\Psi_h(\varphi_h)$ is

$$\lim_{\eta \to \infty} \Psi_h(\eta) = -(2\ln 2 + 1) + \ln \eta + 3q \qquad (3.24)$$

Thus, $\lim_{a \to \infty} \Psi_h(a) + C_h^a = -(2\ln 2 + 1) = -2.38629$. This limiting value is only slightly different from $C_h^{a=5}$. The approximation we shall use for Ψ_h is, therefore,

$$\Psi_h(\eta \geq 5) \approx \ln \eta + 3q\left(1 + \frac{1}{9}q^4 + \frac{1}{45}q^4\right) - 2.38654 \qquad (3.25)$$

Table 7 compares the exact values of Ψ_h with the values given by (3.25).

Table 7. Comparison of the exact values of Ψ_h with the approximate values computed from Eq. (3.25)

η	Ψ_h^{exact}	$\Psi_h^{Eq.(3.25)}$
5	0.91520	0.915120
6	1.0041	1.0040
8	1.1535	1.1533
10	1.2768	1.2766
25	1.8455	1.8452
50	2.8619	2.8617

4. Other surface layer relationships

Drag and heat transfer coefficient methods are common and convenient formulations for calculating momentum, heat, and moisture fluxes. By definition of the drag (C_D) and heat transfer (C_H) coefficients, we have

$$\text{momentum flux} = -\rho C_D U^2 \tag{4.1a}$$

$$\text{sensible heat flux} = -\rho C_p C_H U \Delta\theta \tag{4.1b}$$

$$\text{latent heat flux} = -\rho \mathcal{L} C_q U \Delta q . \tag{4.1c}$$

From the results in the previous section, it follows that

$$C_D = u_*^2 / U^2 = k^2 / F_m^2 \tag{4.2a}$$

and

$$C_q = C_H = u_* \theta_* / U \Delta\theta = k^2 / b F_m F_h . \tag{4.2b}$$

The coefficients C_D and C_H are non-negative and vary smoothly with increasing instability. Table 8 gives values of C_D and C_H for the Businger - GUST relations for various values of $-z/L$ and z/z_o. C_D and C_H are increasingly sensitive to changes in stability for increasingly larger roughness heights.

The values K_m and K_h, the eddy diffusion coefficients for momentum and heat, can be determined from the relations

Table 8. Drag (C_Dx10^3) and heat transfer (C_Hx10^3) coefficients as a function of $-z/L$ and z/z_o for the Businger - GUST profile relations

$-z/L$	$z/z_o = 5\times10^2$		$z/z_o = 5\times10^3$		$z/z_o = 5\times10^4$	
	C_D	C_H	C_D	C_H	C_D	C_H
0.0	3.17	4.29	1.69	2.28	1.05	1.41
0.1	3.50	4.74	1.81	2.45	1.11	1.50
0.5	4.32	5.81	2.11	2.84	1.24	1.68
1.0	5.01	6.70	2.33	3.13	1.34	1.81
2.5	6.48	8.58	2.77	3.69	1.53	2.04
5.0	7.89	10.50	3.25	4.31	1.72	2.29
10.0	11.30	14.60	3.92	5.16	1.97	2.61
25.0	19.30	24.40	5.26	6.87	2.41	3.18
50.0	33.40	41.10	6.89	8.91	2.88	3.79
100.0	73.70	86.20	9.49	12.10	3.53	4.61

$$u_*^2 = K_m \frac{\partial U}{\partial z} \tag{4.3a}$$

$$u_* \theta_* = K_h \frac{\partial \theta}{\partial z} \tag{4.3b}$$

which can be rewritten as

$$K_m = k u_* z / \varphi_m (z/L) \tag{4.4a}$$

$$K_h = k u_* z / b \varphi_h (z/L). \tag{4.4b}$$

The ratio $\alpha(\xi) = K_h/K_m = \varphi_m/b\varphi_h$, the surface layer inverse Prandtl number, is 1.35 for near-neutral conditions for the Businger relations. As $-\xi$ increases, α increases as $\sim$ 2.06 $(-\xi)^{1/4}$ for large $-\xi$. On the other hand, the Businger - GUST relations show no such increase with $-\xi$, and α approaches the limiting value of 1.14 for $-\xi \to \infty$ (see Table 9). The Dyer - GUST relations yield α = 1.0 for near-neutral conditions and differ only slightly from unity for all values of $-\xi > 0$. All relations of the GUST form must have $\alpha \to \alpha_\infty =$ const and $K_{m,h} \sim z^{4/3}$ for $-\xi \to \infty$. Thus, $\alpha(\xi)$ cannot increase without bound for $-\xi \gg 1$.

Table 9. The inverse Prandtl number α, where $\alpha = K_h/K_m$ for the Businger-GUST relations (α_{BG}) and the Businger profile relations (α_B)

	α_{BG}	α_B
0.0	1.35	1.35
0.1	1.34	1.48
0.5	1.25	1.86
1.0	1.22	2.14
2.5	1.18	2.63
5.0	1.17	3.10
10.0	1.16	3.68
25.0	1.15	4.61
50.0	1.15	5.48
100.0	1.14	6.52

As noted in Section 3, the 'inverse problem' in which ξ is computed from the bulk Richardson number R_B,

$$\xi = R_B F_m^2 / bF_h \tag{4.5}$$

requires an iterative solution since F_m and F_h depend nonlinearly upon ξ and ξ_o. For near-neutral conditions; however,

$$\xi \approx \frac{R_B}{b} \ln\left(\frac{h}{z_o}\right) \qquad (4.6)$$

For $-\xi \gg 1$, it is possible to derive a limiting expression for (4.5). From the GUST relations (3.20, 24), it follows that for $-\xi \gg 1$,

$$F_m \rightarrow \ln\left(\frac{h}{z_o}\right) - \ln \eta + C_m \qquad (4.7a)$$

$$F_h \rightarrow \ln\left(\frac{h}{z_o}\right) - \ln \eta + C_h \qquad (4.7b)$$

in which $C_m \approx 2.65$, $C_h \approx 2.39$. Using $\eta_{m,h} = -\gamma_{m,h}\,\xi$, we see that F_m and F_h become

$$F_m \rightarrow \ln(-L/z_o) + a_m \qquad (4.8a)$$

$$F_h \rightarrow \ln(-L/z_o) + a_h \qquad (4.8b)$$

in which

$$a_{m,h} = C_{m,h} - \ln \gamma_{m,h}. \qquad (4.9)$$

Eqs. (4.8) allow (4.5) to be approximated by

$$z_o/L \approx \frac{g\, z_o\, \Delta\theta}{b\,\bar{\theta}\, U^2} \frac{[\ln(-L/z_o) + a_m]^2}{[\ln(-L/z_o) + a_h]}$$

or

$$\hat{R}_B \hat{L} \approx \frac{\ln(-\hat{L}) + a_h}{[\ln(-\hat{L}) + a_m]^2} \qquad (4.10)$$

in which $\hat{L} = L/z_0$ and $\hat{R}_B = g z_0 \Delta\Theta / b\bar{\Theta} U^2$. We see that h drops out as a relevant parameter. Eq. (4.10) is a relatively simple nonlinear relation for which the right-side is a slowly-varying function of $\hat{L}$. Thus, a crude approximation is $\hat{R}_B \hat{L} \approx C$ = const. A better approximation is

$$\hat{R}_B \hat{L} \approx \frac{[\ln(-c/\hat{R}_B) + a_h]}{[\ln(-c/\hat{R}_B) + a_m]^2}$$

or

$$\hat{L} = \frac{1}{\hat{R}_B} \frac{[a_h + \ln c - \ln(-\hat{R}_B)]}{[a_m + \ln c - \ln(-\hat{R}_B)]^2} . \tag{4.11}$$

A more accurate result over a wider range of R can be calculated from

$$\hat{L} = \frac{1}{\hat{R}_B} \frac{(a_0 + a_1 x)}{1 + b_1 x + b_2 x^2} \tag{4.12}$$

in which $x = \ln(-\hat{R}_B)$ and a_0, a_1, b_1, b_2 are calculated using rational fraction methods. When this expression is combined with an approximation valid for mildly unstable conditions, the result is a method that provides accurate, efficient noniterative estimates of L. This method provides a simple method of computing surface fluxes of momentum, heat, and moisture in numerical weather prediction models that have a thin ($\approx$ 25–100m) lower layer. The full solution to the inverse problem will be given in a future report.

5. Conclusions

A General Unified Similarily Theory (GUST) unites the standard empirical flux-profile relations for the unstable surface planetary boundary layer with the predictions of free-convection theory. The basic relations consist of the well-known 'KEYPS' quartic equation for wind and a new cubic equation for temperature. The positive roots of the quartic and cubic equation represent the nondimensional wind shear (φ_m) and temperature gradient (φ_h), respectively. Although the use of the GUST relations appears computationally inefficient compared, for example, to the standard Businger or Dyer flux relations, computationally simple rational fraction and asymptotic relations are developed that closely approximate the exact values of the GUST $\varphi_m(z/L)$ and $\varphi_h(z/L)$. For slightly unstable regimes φ_m and φ_h approximate the standard '$-\frac{1}{2}$' and '$-\frac{1}{4}$' power profile expressions. As instability increases ($-z/L >> 1$), φ_m and φ_h approach the $z^{-1/3}$ power law predicted by free-convection theory.

One of the prime uses of flux-profile relations is in the solution of an 'inverse problem' that arises in numerical weather prediction models. The inverse problem consists of the determination of h/L, u_*, θ_* and the surface fluxes of momentum and heat from the known values of $U(h)$, $\Delta\theta = \theta(h) - \theta(z_0)$ and z_0. The surface fluxes of momentum and heat are given by $-\rho u_*^2$ and $-\rho C_p u_* \theta_*$. The solution for h/L requires the iteration of a nonlinear expression containing the factors F_m and F_h which involve the integrals of $\varphi_m(\xi)/\xi$ and $\varphi_h(\xi)/\xi$ from $\xi_0 = z_0/L$ to $\xi = h/L$, and also $R_B (= gh\Delta\theta/\bar{\theta}U^2)$, the bulk Richardson number. As is the case with φ_m and φ_h, F_m and F_h can be approximated by rational fractions and asymptotic expressions that are no more computationally burdensome than the corresponding forms of the Businger and Dyer F_m and F_h. For the limiting case of extreme instability, the quantities h/L and $R_B = gh\Delta\theta/\bar{\theta}U^2$ drop out (h becomes an irrelevant factor) and are replaced by $\hat{L} = L/z_0$ and $\hat{R}_B = gz_0\Delta\theta/\bar{\theta}U^2$. This simplification leads to a nonlinear equation whose solution can be readily approximated by a rational fraction.

APPENDIX: NOTATION

a = specific value of η ; $a_{m,h}$ = constants in asymptotic relations

b = constant in nondimensional temperature gradient (= neutral Prandtl number)

c = constant in nondimensional humidity gradient; $C^{a}{}_{m,h}$ constants of integration; C_D = drag coefficient; $C_{H,q}$ = heat and moisture transfer coefficients; C_p = specific heat of air at constant pressure

f = arbitrary function; $F_{m,h}$ = integrals involving momentum and temperature profiles

g = acceleration due to gravity; GUST = General Unified Similarity Theory

h = arbitrary height within the surface layer such that $h \gg z_o$.

$I_{m,h}(\eta_{m,h})$ = integrals involved in the asymptotic expressions for $\Psi_{m,h}$

j = iteration index; x - coordinate index

k = von Kármán's constant; $K_{m,h}$ = eddy diffusion coefficients for momentum and heat

L = Obukhov length scale; degree of polynomial in the numerator of a rational fraction; $\hat{L} = L/z_o$; $\mathcal{L}$ = latent heat of vaporization

M = degree of polynomial in the denominator of a rational fraction

N = number of points ($N = L+M+1$) that can be fit with a rational fraction

$Q_{h,o}$ = factors in the integrated forms of the flux-profile relations; q = specific humidity; $q = \eta^{-\frac{1}{3}}$; q_* = turbulent scaling specific humidity

R_B = bulk Richardson number ($= gh\Delta\theta/\bar{\theta}U^2$); $\hat{R}_B$ = modified bulk Richardson number ($= z_o R_B/bh$); $R_{h,o}$ = factors in the integrated forms of the flux-profile relations

U = wind speed within the surface layer at height z or h; u_* = "friction" velocity

x_j = j^{th} point on the x-axis

y_j = j^{th} value of $f(x_j)$

z = arbitrary height; z_o = roughness length (height)

$\alpha(\zeta) = K_h(z,\zeta)/K_m(z,\zeta)$ (= inverse Prandtl number)

γ = generic for $\gamma_{m,h}$; $\gamma_{m,h}$ = constants multiplying z/L in profile relations for wind and temperature

η = generic for $\eta_{m,h}$; $\eta_{m,h} = -\gamma_{m,h} z/L = -\gamma_{m,h} \xi$

θ = potential temperature; $\theta_o = \theta(z = z_o)$; $\bar{\theta}$ = reference or mean potential temperature; θ_* = turbulent scaling temperature

ρ = density of air

$\varphi_{m,h,q}$ = nondimensional "universal" gradients of wind speed, temperature, and humidity

$\psi_{m,h}$ = integrated forms of nondimensional gradients of wind and temperature

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